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Updating Large Finite Element Models in Structural Dynamics

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In model updating problems, one tries to find design variable values affecting the input to a numerical or analytical model, such that the model output is well correlated to some target response or experimental data. Because most proposed solution techniques are iterative or require large matrices to be inverted and multiplied, severe obstacles arise when large models, such as, for example, typical industrial finite element models, are at hand. Here we consider finite element model updating for structural dynamic applications and show how the problem with little effort may be projected onto a subspace spanned by, for instance, a few eigenvectors of the initial model. Because the subspace dimension is much smaller than the space spanned by the finite element basis, the subsequent correlation can be conducted speedily, although large changes from the initial design may require the subspace basis to be updated. The proposed method can conveniently be used for substructured models and, in addition, makes interactive design changes possible. A small model problem and an industrial application are used to illustrate the performance of the suggested approach.

Nomenclature

Scalars

f_i	= function of the modal assurance criterion (*MAC* number) for the ith mode pair
MAC_i	= *MAC* number for the ith mode pair
N_a	= number of analytical eigenpairs solved for
N_{af}	= number of degrees of freedom in analytical model
N_d	= subspace dimension
N_{ndv}	= number of design variables
N_s	= number of substructures
N_x	= number of experimental eigenpairs
N_{xf}	= number of transducer locations (experimental degrees of freedom)
w	= weight factor
x_i, x_{li}, x_{ui}	= ith design variable, lower bound, and upper bound
β_{ij}	= coefficients in linear combination to form derivative of ith analytical eigenvector with respect to a design variable
Δx_i	= increment of the ith design variable
η_i	= subspace approximation of λ_{ai}
λ_{ai}	= analytical eigenvalue of mode that has been paired with the ith experimental mode
λ_{xi}	= ith experimental eigenvalue
Π	= objective function to be minimized

Vectors, Matrices, and Operators

$\boldsymbol{D}_{Ki}, \boldsymbol{D}_{Mi}$	= linearized derivatives of structure stiffness and mass matrices, with respect to the ith design variable
$\boldsymbol{e}_i$	= vector with ith element equal to 1, and 0 elsewhere
$\boldsymbol{K}, \boldsymbol{M}$	= structure stiffness and mass matrices
$\boldsymbol{x}$	= vector of design variable values
$\boldsymbol{x}^*$	= design point at which Π is stationary
Φ	= matrix with columns being the basis vectors of a subspace of the analytical model
ϕ_{ai}	= analytical eigenmode that has been paired with the ith experimental mode
$\hat{\phi}_{ai}$	= ϕ_{ai} restricted to the measurement (transducer) degrees of freedom
ϕ_{xi}	= ith experimental eigenmode
ψ_i	= subspace eigenvector, coordinates for approximation of ϕ_{ai}
∇	= gradient operator for scalar function
$(,)$	= scalar product of vectors
$[\,]_i$	= ith component of vector

Subscripts

(i)	= entity pertains to substructure number i
r	= entity has been projected onto a subspace (reduced system)

Superscripts

(i)	= entity pertains to the ith design ($i = 0$ for the initial design)
T	= vector and matrix transpose

I. Introduction

ALTHOUGH the development of the finite element (FE) method, along with access to computers with seemingly ever-growing computational speed and storage capabilities, has made it possible to solve large and complex problems, complicated structures are in practice usually modeled in a simplified manner. Typically, discrete masses, springs, and bar and beam elements are extensively used in structural dynamics, whereas membrane, plate, and shell elements might be added as found necessary. In practice the analyst has to rely on experience and engineering intuition when selecting properties (such as constitutive parameters, beam cross-sectional parameters, magnitudes of discrete masses, etc.), but it is obvious that large modeling errors may be introduced, aside from the usual discretization inaccuracies. Once experimental results are available, however, it might be possible to adjust selected properties so as to make the FE model better reproduce the real behavior in some respects. The solution of this inverse problem is the goal of FE model updating.

Although model updating in principle is applicable in various types of problems, it seems that it is utilized almost exclusively in structural dynamic applications, at least in the context of FE analysis. There are several methods available to formulate and to solve the problem, and surveys are given by Imregun and Visser,[1] Ibrahim,[2] and Mottershead and Friswell.[3] Because model updating is an iterative process and the FE problem has to be solved at least once in each iteration, the cost is usually prohibitive when industrial-sized models are considered. Thus, most published results consider

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small model problems (for example, see Refs. 4–8), although a few attempts to solve industrial models have been reported.[9,10]

In this paper, we present an approach to the model update problem where the FE model is projected onto a small subspace, spanned by a few eigenvectors, in which the correlation to experimental data is performed. The design variables thus calculated (model parameters) immediately pertain to the FE model. Because the subspace dimension typically is several orders of magnitude smaller that the space spanned by the FE basis functions in a typical industrial-sized problem, the suggested approach is appealing when large, and possibly substructured, models are considered. In addition, interactive model updating is made possible, which means that the analysts' experience and intuition can be exploited, reasonable and useful designs are assured, and insight is gained for the particular problem at hand, as noted by Arora and Tseng.[11]

II. Problem Formulation

A. Objective Function and Minimization Problem

To evaluate to what extent a particular design $\boldsymbol{x}$ yields eigenpairs that agree with experimental results, one needs some measurement. This measurement is an objective function Π that reflects the deviation between the FE results, for a constant mesh density, and the real behavior. Hence, if $\Pi(\boldsymbol{x}^{(i)}) < \Pi(\boldsymbol{x}^{(j)})$, we may immediately conclude that the ith design is better than $\boldsymbol{x}^{(j)}$, and it follows that the model correlation may be posed as a minimization problem, viz., find $\boldsymbol{x}^*$ such that

$$\begin{aligned} &\Pi(\boldsymbol{x}^*) \le \Pi(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \\ &x_{li} \le x_i \le x_{ui}, \qquad i = 1, 2, \ldots, N_{\text{ndv}} \end{aligned} \tag{1}$$

where upper and lower bounds on the design variables were introduced.

It should be clear that the selection of the objective function has a profound impact on the problem. Not only does it affect our interpretation of best correlation, but its properties will influence the behavior of the utilized optimization algorithm. Because we do not know the location of the global optimum, an object function that is convex, or at least has a monotone decay, in a region as large as possible, is desirable in case one wants a fully automatic model update procedure. Indeed, an approach to this end has been described,[8] but large models normally require significant deviations from the initial design to conform to stipulated demands, and thus it may be hard to devise a foolproof objective function. Instead a man-in-the-loop optimization strategy seems to be preferable, which is also the experience from industrial applications. In either case, the optimization requires us to solve the FE problem at least once in each iteration to evaluate the objective function value. If this cannot be done fast, interactive model updating is for practical reasons hardly possible.

We have chosen to formulate the objective function in terms of the discrepancy between FE and experimental eigenvalues and mode shapes, viz.,

$$\Pi(\boldsymbol{x}) = \sum_{i=1}^{N_x} \left[w \left(\frac{\lambda_{ai} - \lambda_{xi}}{\lambda_{xi}} \right)^2 + (1 - w) f_i \right], \qquad 0 \le w \le 1 \tag{2}$$

where f_i is a nonnegative function that measures the discrepancy in shapes between the ith experimental mode and the FE mode that has been paired with it. The weight factor w may be used to impose a relative difference in importance between eigenvalue and mode shape deviations because these entities may have been measured with different accuracies. Also note that, in Eq. (2), λ_{ai} is the FE eigenvalue of the mode that has been paired with the ith experimental mode, which is not necessarily the same as the ith FE mode. We have tried several expressions for f_i and found that

$$f_i = f(MAC_i) = \frac{(1 - \sqrt{MAC_i})^2}{MAC_i} \tag{3}$$

where

$$MAC_i = \frac{(\hat{\boldsymbol{\phi}}_{ai}, \boldsymbol{\phi}_{xi})^2}{(\hat{\boldsymbol{\phi}}_{ai}, \hat{\boldsymbol{\phi}}_{ai})(\boldsymbol{\phi}_{xi}, \boldsymbol{\phi}_{xi})} \tag{4}$$

is the modal assurance criterion[12] evaluated for the ith mode pair, has worked best in most situations.[13] Equation (2) measures the discrepancy between N_x pairs of modes, once these have been matched to each other. The actual mode matching is a critical task because erroneous matching frequently impedes an efficient model update. This topic is outside the scope of the current work, but we refer to Ref. 13 or 14 for a detailed account of our approach to this problem.

It is recognized that before comparing an FE mode to an experimental one, the former has to be restricted to the transducer locations or the latter has to be prolonged to the FE degrees of freedom; for example, see Refs. 1, 2, and 15–18. Both approaches are expensive in terms of computational effort and pollute experimental data with analytical results, which are known to be defective (or model correlation would not be considered in the first place). For these reasons, simple truncation of the FE eigenvectors seems to be the most commonly adopted approach.

B. Objective Function Gradient

We use recursive quadratic programming[19] (RQP) as implemented in the software IDESIGN[20] to solve the minimization problem, Eq. (1). To this end, the objective function gradient

$$[\nabla \Pi]_j = \frac{\partial \Pi}{\partial x_j} = \sum_{i=1}^{N_x} \frac{\partial \Pi}{\partial \lambda_{ai}} \frac{\partial \lambda_{ai}}{\partial x_j} + \frac{\partial \Pi}{\partial f_i} \frac{\partial f_i}{\partial MAC_i} \frac{\partial MAC_i}{\partial x_j} \tag{5}$$

is needed. Here,

$$\frac{\partial MAC_i}{\partial x_j} = \sum_{k=1}^{N_{xf}} \frac{\partial MAC_i}{\partial [\hat{\boldsymbol{\phi}}_{ai}]_k} \frac{\partial [\hat{\boldsymbol{\phi}}_{ai}]_k}{\partial x_j} \tag{6}$$

and from the definition in Eq. (4) we get

$$\frac{\partial MAC_i}{\partial [\hat{\boldsymbol{\phi}}_{ai}]_k} = \frac{2MAC_i}{(\hat{\boldsymbol{\phi}}_{ai}, \boldsymbol{\phi}_{xi})} \left\{ [\boldsymbol{\phi}_{xi}]_k - [\hat{\boldsymbol{\phi}}_{ai}]_k \frac{(\hat{\boldsymbol{\phi}}_{ai}, \boldsymbol{\phi}_{xi})}{(\hat{\boldsymbol{\phi}}_{ai}, \hat{\boldsymbol{\phi}}_{ai})} \right\} \tag{7}$$

From Eqs. (5) and (6) it is recognized that the derivatives of the FE eigenvalues and eigenvectors have to be evaluated to calculate the objective function gradient.

C. Eigenpair Derivatives

The calculation of eigenvalue and eigenvector derivatives has been extensively studied and reported in many papers (for example, see Ref. 21) because the concept is important in sensitivity analysis and various other dynamic applications. In this study, the expressions derived by Fox and Kapoor[22] are used: Differentiating the generalized eigenvalue problem with respect to a design variable and using the orthogonalization properties of the eigenvectors, one arrives at

$$\frac{\partial \lambda_{ai}}{\partial x_k} = \boldsymbol{\phi}_{ai}^T \left(\frac{\partial \boldsymbol{K}}{\partial x_k} - \lambda_{ai} \frac{\partial \boldsymbol{M}}{\partial x_k} \right) \boldsymbol{\phi}_{ai} \tag{8}$$

where it is assumed that the eigenvectors have been normalized such that the modal masses are unity.[22]

The mode vector derivative may be expressed as a linear combination of all eigenvectors, i.e.,

$$\frac{\partial \boldsymbol{\phi}_{ai}}{\partial x_k} = \sum_{n=1}^{N_{af}} \beta_{in} \boldsymbol{\phi}_{an} \tag{9}$$

where the coefficients β_{in} are determined using the generalized eigenvalue problem and orthogonalization properties of eigenvectors. Provided that the eigenvectors have been normalized to unit modal masses, one gets[22]

$$\beta_{ij} = \begin{cases} \boldsymbol{\phi}_{aj}^T \left[\left(\dfrac{\partial \boldsymbol{K}}{\partial x_k} - \lambda_{ai} \dfrac{\partial \boldsymbol{M}}{\partial x_k} \right) \Big/ (\lambda_{ai} - \lambda_{aj}) \right] \boldsymbol{\phi}_{ai}, & i \ne j \\ -\dfrac{1}{2} \boldsymbol{\phi}_{ai}^T \dfrac{\partial \boldsymbol{M}}{\partial x_k} \boldsymbol{\phi}_{ai}, & i = j \end{cases} \tag{10}$$

Because the full eigensystem in general is not available and far too expensive to solve for, the summation in Eq. (9) is in practice

over $N \leq N_a \ll N_{af}$ modes, and so the evaluated derivatives become approximations only. For design variables with global range, e.g., a density that pertains to the whole structure, the truncated series may of course provide an accurate estimation of the derivative. However, for a variable with local impact, for instance the magnitude of a discrete mass, one expects the true mode derivative to resemble a Green function; i.e., its value may be large in the vicinity of the design change but small elsewhere. Because it is known that eigenmode series expansions of such functions have very slow convergence,[23] one must expect the truncated version of Eq. (9) to yield crude approximations for most design variables, which also has been observed in numerical experiments.[24,25]

It should be observed that Eqs. (8) and (10) require the FE eigenvalues to be distinct. Although it is possible to calculate derivatives for repeated modes (for example, see Refs. 26–28), these become directional derivatives only[29] and are thus not useful when we need to evaluate the objective function gradient before the search direction in the minimization. In a numerical solution, eigenvalues are not likely to become exactly identical, but even a small difference may cause numerical errors due to cancellation. It is, however, easy to avoid such design points by adjusting the step length, for any given search direction, in the iterative solution of Eq. (1).

D. Linearization of Matrix Derivatives

From Eqs. (8) and (10), it is seen that the derivatives of the structure stiffness and mass matrices, with respect to the design variables, must be available. Naturally, analytical expressions for these entities may be developed, but a new programming effort would be required each time a new type of design variable is introduced. In addition, we would be restricted to a FE code to which we have access to the source code and to the use of element types for which explicit matrix derivatives have been developed. By adopting linearized matrix derivatives, viz., first-order Taylor expansions at the current design point,

$$\boldsymbol{D}_{Kk} = \frac{\boldsymbol{K}\left(\boldsymbol{x}^{(i)} + \Delta x_k \boldsymbol{e}_k\right) - \boldsymbol{K}\left(\boldsymbol{x}^{(i)}\right)}{\Delta x_k}$$
$$\boldsymbol{D}_{Mk} = \frac{\boldsymbol{M}\left(\boldsymbol{x}^{(i)} + \Delta x_k \boldsymbol{e}_k\right) - \boldsymbol{M}\left(\boldsymbol{x}^{(i)}\right)}{\Delta x_k} \tag{11}$$

we avoid these limitations. It is observed that the expressions are exact in case the matrices are linear with respect to the kth design variable.

Note that the evaluation of approximate matrix derivatives according to Eq. (11) does not involve any additional problem solution, but it suffices to assemble and to save the appropriate system matrix for each design variable increment, which is a minor computational effort as compared with the solution of the eigenproblem.

III. Model Size Reduction

A. Preliminaries

Although the previously described material in theory is sufficient to conduct model correlation in structural dynamics, some practical obstacles remain. The solution of the minimization problem of Eq. (1) requires the eigenproblem to be solved at least once in each iteration, which for most real problems leads to prohibitive computational costs. Another less attractive feature in the current setting is that many industrial FE models for practical reasons are substructured. This means that the solution of the generalized eigenproblem typically requires several program executions, and so substructuring makes iterative model updating not only expensive in terms of CPU time but also tedious, complex, and error prone unless it can by automated.

In this section we propose that the FE model can be represented in a subspace that well represents the considered eigenpairs but still is substantially smaller than the space spanned by the FE basis functions. If the subspace with reasonable accuracy approximates the eigenpairs subsequent to a moderate design change, it is possible to solve the minimization problem in the subspace, and hence considerable computational savings are achieved. Naturally, substantial deviations from the initial design may require the subspace basis to be updated, but as long as the changes are reasonable and the linearized matrix derivatives are appropriate [see Eq. (11)], one may retain the subspace basis, and solutions of the full eigenproblem are avoided.

It should be clear that the approach suggested next is not dependent on a particular choice of objective function or even our particular application but could be useful in any situation where one is requested to repeatedly solve the generalized eigenvalue problem subsequent to design changes.

B. Subspace Formulation

Because we want an accurate representation of the eigenpairs that matches the measured quantities and, in addition, the FE eigenpairs are available for the initial design, it seems natural and highly efficient to use the eigenvectors ϕ_{ai}, or a subset of these, as a basis for a subspace of dimension N_d. Thus, $N_d \leq N_a \ll N_{af}$, and the selected N_d may influence how many eigenpairs are initially solved for. For the purpose of presentation, let Φ be an $N_{af} \times N_d$ matrix with the ith basis vector for the selected subspace in the ith column. Furthermore, let us use superscript (j) to refer an entity to the jth design, and in particular (0) for the initial design. Thus, we write

$$\boldsymbol{K}^{(j)} = \boldsymbol{K}^{(0)} + \sum_{k=1}^{N_{\text{ndv}}} \boldsymbol{D}_{Kk}\left(x_k^{(j)} - x_k^{(0)}\right)$$
$$\boldsymbol{M}^{(j)} = \boldsymbol{M}^{(0)} + \sum_{k=1}^{N_{\text{ndv}}} \boldsymbol{D}_{Mk}\left(x_k^{(j)} - x_k^{(0)}\right) \tag{12}$$

and

$$\phi_{ai}^{(j)} \approx \Phi \psi_i^{(j)} \tag{13}$$

Substituting the transformation equation (13) into the generalized eigenproblem and premultiplying by Φ^T yields

$$\left[\boldsymbol{K}_r^{(j)} - \eta_i \boldsymbol{M}_r^{(j)}\right]\psi_i^{(j)} = \boldsymbol{0}_r, \qquad i = 1, 2, \ldots, N_d \tag{14}$$

where

$$\boldsymbol{K}_r^{(j)} = \Phi^T \boldsymbol{K}^{(j)} \Phi, \qquad \boldsymbol{M}_r^{(j)} = \Phi^T \boldsymbol{M}^{(j)} \Phi \tag{15}$$

Because the subspace eigenvector $\psi_i^{(j)}$ in general provides only an approximation to the full model vector $\phi_{ai}^{(j)}$ [cf. Eq. (13)], η_i is just an approximation of the true eigenvalue

$$\lambda_{ai} \approx \eta_i \tag{16}$$

Substituting Eq. (12) into Eq. (15), we get

$$\boldsymbol{K}_r^{(j)} = \boldsymbol{K}_r^{(0)} + \sum_{k=1}^{N_{\text{ndv}}} \boldsymbol{D}_{rKk}\left(x_k^{(j)} - x_k^{(0)}\right)$$
$$\boldsymbol{M}_r^{(j)} = \boldsymbol{M}_r^{(0)} + \sum_{k=1}^{N_{\text{ndv}}} \boldsymbol{D}_{rMk}\left(x_k^{(j)} - x_k^{(0)}\right) \tag{17}$$

with

$$\boldsymbol{D}_{rKk} = \Phi^T \boldsymbol{D}_{Kk} \Phi, \qquad \boldsymbol{D}_{rMk} = \Phi^T \boldsymbol{D}_{Mk} \Phi \tag{18}$$

being the projections of the linearized matrix derivatives. Although the matrices $\boldsymbol{D}_{rKk}$ and $\boldsymbol{D}_{rMk}$ may be established according to Eqs. (11) and (18), we point out that considerable computational effort may be saved if the projected derivatives are evaluated as

$$\boldsymbol{D}_{rKk} = (1/\Delta x_k)\left[\Phi^T \boldsymbol{K}\left(\boldsymbol{x}^{(0)} + \Delta x_k \boldsymbol{e}_k\right)\Phi - \boldsymbol{K}_r^{(0)}\right]$$
$$\boldsymbol{D}_{rMk} = (1/\Delta x_k)\left[\Phi^T \boldsymbol{M}\left(\boldsymbol{x}^{(0)} + \Delta x_k \boldsymbol{e}_k\right)\Phi - \boldsymbol{M}_r^{(0)}\right] \tag{19}$$

because this involves subtractions with much smaller matrices.

Hence, once the structure matrices and their respective linearized derivatives have been projected onto the selected subspace, projected matrices at any design $\boldsymbol{x}^{(j)}$ are obtained according to Eq. (17),

and the subspace eigenproblem, Eq. (14), may be solved. Subsequently, one obtains approximations to the eigenvalues and eigenvectors of the original FE model according to Eqs. (16) and (13), respectively.

Note that the complete eigensystem, Eq. (14), is solved for. For the initial design the matrices are diagonal and the solution is immediately available. With reasonable design changes one may expect $\boldsymbol{K}_r^{(j)}$ and $\boldsymbol{M}_r^{(j)}$ to be diagonally dominant, in which case there are iterative solution methods, such as the Jacobi transformation method (for example, see Ref. 30), that possess quadratic convergence properties; at any rate, solving the full system of Eq. (14) is much faster than extracting the first few eigenpairs from a typical industrial-sized FE model.

C. Optimization in Subspace

Because the objective function includes the eigenvalues and *MAC* numbers, we still need to calculate their respective derivatives. Considering the eigenvalues first, we note that because of the approximation of Eq. (16) one can use Eq. (8) to calculate the derivatives of η_i rather than λ_i; i.e., we employ

$$\frac{\partial \lambda_{ai}}{\partial x_k} \approx \frac{\partial \eta_i}{\partial x_k} = \psi_i^T \left(\frac{\partial \boldsymbol{K}_r}{\partial x_k} - \eta_i \frac{\partial \boldsymbol{M}_r}{\partial x_k} \right) \psi_i \tag{20}$$

where the subspace matrix derivatives of course are obtained as

$$\frac{\partial \boldsymbol{K}_r}{\partial x_k} \approx \boldsymbol{D}_{rKk}, \qquad \frac{\partial \boldsymbol{M}_r}{\partial x_k} \approx \boldsymbol{D}_{rMk} \tag{21}$$

compare Eq. (17). Notice the substantial saving in computational effort that is achieved by calculating the derivatives according to Eq. (20) rather than employing Eq. (8).

As for the eigenvectors, Eq. (13) yields

$$\frac{\partial \phi_{ai}}{\partial x_k} \approx \Phi \frac{\partial \psi_i}{\partial x_k} \tag{22}$$

because Φ is constant. Thus, we calculate the subspace vector derivatives in analogy with Eqs. (9) and (10) as

$$\frac{\partial \psi_i}{\partial x_k} = \sum_{n=1}^{N_d} \beta_{in} \psi_n \tag{23}$$

with

$$\beta_{ij} = \begin{cases} \psi_j^T \dfrac{(\boldsymbol{D}_{rKk} - \eta_i \boldsymbol{D}_{rMk})}{\eta_i - \eta_j} \psi_i, & i \neq j \\ -\frac{1}{2} \psi_i^T \boldsymbol{D}_{rMk} \psi_i, & i = j \end{cases} \tag{24}$$

and obtain approximations to the requested full model mode derivatives according to Eq. (22). It is observed that the series of Eq. (23) is not truncated here, but the evaluated derivative becomes exact apart from numerical errors. Naturally, a form of truncation is still applied in Eq. (22), but the error thus introduced depends on the selected subspace basis rather than on approximated formulas. Note that it suffices to calculate the derivative vector components that pertain to measured degrees of freedom only, and so in practice we remove the rows in Φ that correspond to unmeasured quantities before carrying out the matrix by vector multiplication in Eq. (22).

As with the eigenvalue derivatives, the subspace approach gives significant time savings when calculating the vector derivatives. However, the evaluation of the mode derivatives typically is the most time-consuming part of the subspace optimization.

D. Substructuring

We wish to emphasize that substructuring the FE model does not impede the outlined subspace method but actually somewhat reduces the computational effort to project the matrices. Rather than projecting the assembled structure stiffness matrix onto the subspace to obtain $\boldsymbol{K}_r$ according to Eq. (15), we project the substructure matrices

$$\boldsymbol{K}_{r(s)} = \Phi_{(s)}^T \boldsymbol{K}_{(s)} \Phi_{(s)}, \qquad s = 1, 2, \ldots, N_s \tag{25}$$

after which the desired reduced matrix is obtained by summation:

$$\boldsymbol{K}_r = \sum_{s=1}^{N_s} \boldsymbol{K}_{r(s)} \tag{26}$$

Naturally the same scheme is used for the mass matrix.

It is recognized that substructuring usually involves a reduction in size by so-called master–slave relationships. Because slave variables are condensed from the substructure matrices, the assembled structure matrix implicitly embraces an assumption about the behavior of the removed variables in terms of the retained degrees of freedom, and projecting this matrix to a subspace could have obscure physical meaning. Thus, the approach outlined earlier may be the only reasonable one. The implication should be noted carefully, however: The FE model updating will aim at making the complete model reproduce the experimental data. Hence, discretization errors introduced by the imposed master–slave relationship are not considered.

In establishing the projections of the linearized matrix derivatives, one typically saves quite a lot of computational effort because a design variable normally will affect a single substructure matrix only. Hence, it suffices to construct and to project the affected substructure matrix.

E. Subspace Dimension and Basis Regeneration

In the foregoing we have described an approach where correlation of a large FE model to experimental data is conducted in a small subspace, spanned by, for instance, eigenvectors pertaining to the original design. Because this procedure is an approximation only, it is advisable to view the projection as an outer loop in an algorithm in which the iterative solution of the minimization problem in the subspace constitutes an inner loop. Naturally, subsequent to the subspace optimization, the full eigenproblem should be solved so as to evaluate the most recent approximations provided by Eqs. (13) and (16). Should the deviation from experimental values still be too large, the newly obtained eigenvectors may be used to project the new exact structure matrices onto a subspace thus updated, i.e., another pass through the outer loop is taken.

The question of when to abandon the inner loop iterations is more difficult to answer in a general manner. There are two issues to take into account in this context: first, when the subspace basis no longer provides a reasonable approximation of the complete model, and second, when the linearized matrix derivatives [Eq. (11)] are no longer valid. The regions of validity (in design space) for both these approximations obviously depend on the selection of design variables and, thus, need to be established for the individual problem at hand. Note, though, that the simple bounds for the design variables [cf. Eq. (1)] can be adjusted easily for each subspace so as to ensure that the design does not stray too far away from the design for which the basis and matrix derivatives were established.

The choice for subspace dimension N_d obviously depends on the number of experimental eigenpairs that are considered, but it is also desirable that the subspace resolve the change in the FE model behavior due to changes in design. Although it may be tempting to use a fairly large subspace dimension in an endeavor to improve the approximation of the modified design, our experience is that such an approach is not very rewarding. As will be seen, the computational effort, which is mainly due to the evaluation of the vector derivatives, grows rapidly, whereas little is gained in the accuracy of the eigenmode approximation. Typically, a design variable variation has a rather local impact on the model, and so one could expect the change in an eigenmode to resemble a Green function for the vibration problem. An eigenfunction series expansion of such a Green function is known to converge utterly slowly,[23] which may explain why it is not worthwhile to improve the subspace approximations by adopting a large number of basis vectors. Instead, two other considerations should govern the selection for an appropriate basis, as follows.

First, we note that, if the rigid-body modes are present in the FE model, all other eigenmodes will be centered about the center of gravity (c.g.) of the structure, in the sense that the c.g. will not experience any acceleration in the absence of external forces. In case the location of the c.g. is changed, e.g., as a result of a design

modification, all eigenmodes will change accordingly; the only way to express this offset is to add rigid-body modes to the original mode shape. Therefore, in case there are design variables that allow for a change in mass distribution, it is important that all present rigid-body eigenmodes are appended to the subspace basis.

Second, one must take experimentally measured quantities into account when selecting the subspace dimension; i.e., the subspace basis must include all FE modes that correspond to some experimental mode. In this context, one must observe inappropriate mode pairing and missing eigenpairs because the first N_x FE eigenpairs (not counting any rigid-body modes) are not necessarily matched to the experimental data, and in addition the mode pairing may change due to design changes. Hence, the subspace basis must be large enough to ensure that the experimental modes are sufficiently well represented. For this reason it is advisable to include many more FE modes in the subspace basis than there are experimentally determined modes. Some guidance in this matter might be obtained from the eigenvalues. For instance, if it can be judged that the maximum error in a FE eigenvalue is 100%, corresponding to about 40% error in vibration frequency, then it may be expedient to use all eigenmodes with eigenvalues less than twice the highest experimental eigenvalue for the subspace basis.

F. Computational Outline

The principal steps in the suggested subspace approach are summarized in the following procedural flowchart. Note that the method is not dependent on the particular objective function described in Sec. II but may be feasible in any situation where eigenmodes are available and the generalized eigenvalue problem has to be solved repeatedly.

Start outer loop:
  Solve generalized eigenvalue problem for full model.
  If converged, then quit; else:
  Project substructure matrices onto selected subspace
    [Eq. (25)] and obtain projected structure matrices by
    summation [Eq. (26)].
  For each design variable: $i = 1, 2, \ldots, N_{\text{ndv}}$.
    Increment design variable i by Δx.
    Establish new stiffness and/or mass matrices for affected
      substructures.
    Project new matrices onto subspace and compute linearized
      derivatives [Eq. (19)].
  Next design variable i.
  Inner loop: Minimize objective function in subspace.
    [At any design, projected structure matrices are
      calculated from Eq. (17).
    Solving the subspace eigenvalue problem, Eq. (14), we
      obtain approximations to the eigenpairs of the full model
      from Eqs. (13) and (16).]
End outer loop.

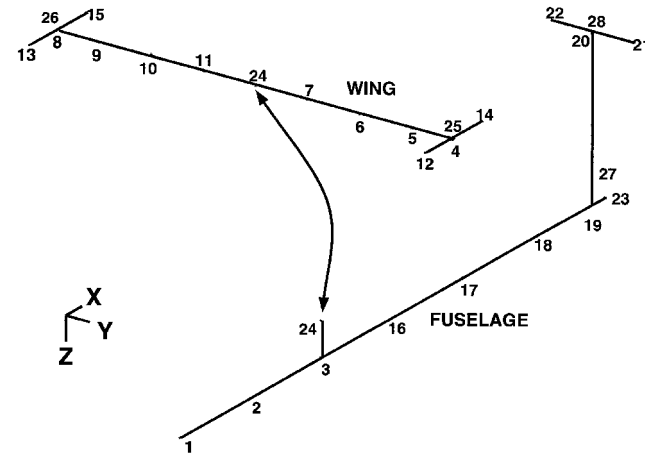

Fig. 1 Substructured FE model of the GARTEUR test problem; numbers indicate node positions.

Imregun and Visser,[1] Ibrahim,[2] and Mottershead and Friswell[3] for details.

We remark that in both examples the target models are sufficiently far away from the respective initial models for mode crossings to occur, and mode pairing is performed in each iteration when minimizing the objective function. Detailed descriptions of the problems may be found in Ref. 13; the mode pairing approach is also detailed in Ref. 14. All computations were done on a Hewlett Packard 9000/735 workstation.

A. GARTEUR: A Small Model Problem

The first test problem concerns a small Group for Aeronautical Research and Technology in Europe (GARTEUR) model. This frame structure is an assemblage of beams and resembles an airplane, although its mass is only about 50 kg. The FE model, shown in Fig. 1, contains 28 nodes with six degrees of freedom in each node and has been substructured so as to verify that our approach to model updating properly handles this.

Altogether 11 design variables were used; these relate to magnitudes of discrete masses at the wing tips and at the horizontal stabilizer, mass density of the fuselage, torsional stiffnesses of the wings, and second moments of inertia for the wings and a part of the fuselage. All variables were normalized with respect to their respective initial values, and target data were generated by

$$\boldsymbol{x}^* = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T \tag{27}$$

whereas the initial design was chosen to be

$$\boldsymbol{x}^{(0)} = [1.1096 \quad 0.6667 \quad 1.1111 \quad 0.7999 \quad 1.1429 \quad 0.8333 \quad 2.0000 \quad 0.6667 \quad 0.5714 \quad 1.2000 \quad 1.2000]^T \tag{28}$$

IV. Sample Problems

Earlier we suggested a method to solve the updating problem for large FE models by approximating the behavior in a subspace spanned by some of the calculated eigenvectors. The intentions with the following two sample problems are to illustrate that the method works with substructured models, yields substantial savings in computational time, and converges to the exact solution. To attain the latter goal, it is necessary to ensure that there is such a thing as an exact solution, which has been accomplished by generating the target model, i.e., experimental data, by perturbed FE models. In a problem with real experimental data, there is, in general, no true solution, but one searches for some improved model; the possibility to succeed depends on a number of factors, such as transducer locations, mode pairing method, selection of objective function, selection of design variables, etc. Because these topics are outside the scope of the current presentation, we refer to the surveys by

The optimization was with respect to the first nine nonzero eigenpairs, and 29 sensor locations were used to measure the eigenmodes. We use subspace dimension 56, which includes the six rigid-body modes, because some design variables affect the mass distribution. The objective function equation (2) with $w = 0.9$ was used in solving the minimization problem. The changes of eigenvalues and *MAC* numbers during 36 iterations are shown in Fig. 2. Because further iterations did not yield any progress, it was decided to solve the full eigensystem at this stage and thereby regenerate the subspace basis. Minimization in the new subspace was done with $w = 0.5$ because it seemed as if the *MAC* numbers for two of the modes tended to stray off (cf. Fig. 2). Now 250 iterations were done, and the results for eigenvalues and *MAC* numbers are displayed in Fig. 3: The subspace approximations of the eigenvalues are identical to the target values, and the worst *MAC* number is 0.9967. Because the subspace is only an approximation to the full model, the eigenproblem of the

Table 1 Eigenvalues for the GARTEUR test problem[a]

Mode number	Target model eigenvalue, s^{-2}	Eigenvalue error, %				
		FEM(0)	SS(1)	FEM(1)	SS(2)	FEM(2)
1	594	−11.27	−1.06	−1.06	0.00	0.00
2	8,151	−24.61	−0.64	−0.64	0.00	0.00
3	34,013	25.58	−0.22	−0.23	0.00	0.00
4	39,196	−2.33	0.26	0.23	0.00	0.00
5	47,817	35.97	0.27	0.24	0.00	−0.05
6	51,595	31.54	0.10	0.09	0.00	−0.06
7	63,875	−8.53	0.86	0.84	0.00	0.00
8	74,046	−18.93	1.04	1.03	0.00	0.00
9	155,766	−14.58	0.26	0.25	0.00	0.00

[a]FEM(i) pertains to the FE model after the ith subspace update ($i = 0$ for the initial model), whereas SS(i) is the estimate in the subspace at the end of the ith update.

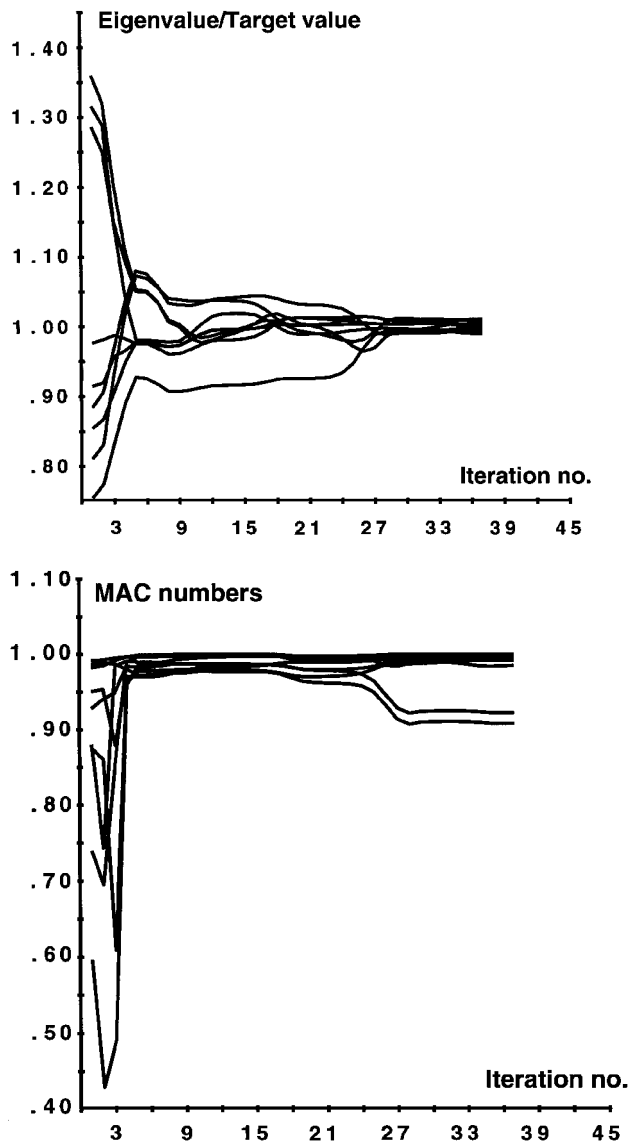

Fig. 2 Convergence of the nine FE eigenpairs to target values during iterations in the initial subspace.

FE model was solved once again to verify the results. Eigenvalues 5 and 6 differed from their target values by 0.05 and 0.06%, respectively, whereas the others were exact; the worst *MAC* number remained at 0.9967. In Table 1 we list the various eigenvalues during the model update.

B. SAAB 340AEW: An Industrial Example

A FE discretization of SAAB 340AEW, as shown in Fig. 4, is considered next. It consists of three substructures: the two (asymmetrical) nacelles, one of which is illustrated, and the remaining part of the structure. There are approximately 12,000 degrees of freedom in the model. Model updating is done with 70 design variables, all of which are normalized to their respective initial values. The target model was obtained by changing design variable values up to ±40%, and 30 eigenmodes were measured at 135 transducer locations.

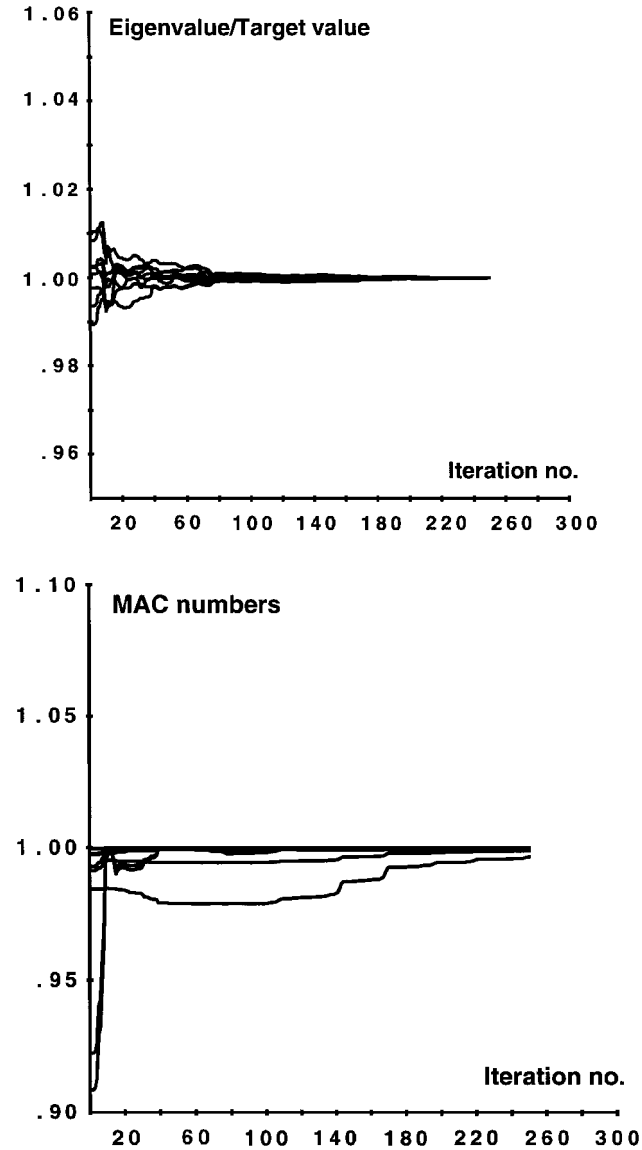

Fig. 3 Convergence of the nine FE eigenpairs to target values during iterations in the updated subspace.

Model updating was carried out with three different subspace dimensions, viz., those spanned by the first 100, 150, and 200 eigenvectors of the initial FE model. Because some design variables affect the mass distribution, the six rigid-body modes were included in the subspaces. It took 11, 29, and 60 s of CPU time per iteration with the RQP method[20] to solve the minimization problem in the respective subspaces. The bulk of the computational effort stems from the calculation of mode shape derivatives, whereas the solution time for the subspace eigenproblem is negligible in all cases. As a comparison to these times, we mention that solving the full FE eigenproblem for the first 150 modes, using the Lanzos method in UAI/NASTRAN, took 393 s. Apart from the differences in execution time, optimizations with subspace dimensions 100, 150, and 200 did not yield any significant differences in the results. Because of this, the iteration times, and the fact that the analysis subspace basis should be updated subsequent to minimization, we tend to favor a small, or at least reasonably sized, subspace dimension.

The subsequent results pertain to model updating with 100 eigenvectors of the initial FE model used as basis vectors for the analysis

Table 2 Target model eigenvalues and errors in the initial and updated SAAB 340AEW FE model

Mode number	Target model eigenvalue, s^{-2}	Eigenvalue error, %		*MAC* number	
		Initial	Updated	Initial	Updated
1	41	0.77	0.17	0.99999	0.99999
2	67	1.42	−0.04	0.99997	1.00000
3	94	0.44	−0.08	0.99993	0.99998
4	134	0.66	0.31	0.99993	0.99997
5	156	0.57	0.25	0.99990	0.99996
6	238	−0.03	−0.24	0.99894	0.99993
7	777	−5.27	0.40	0.99897	0.99999
8	1,140	−1.07	0.11	0.97082	0.99920
9	1,210	2.60	0.67	0.96997	0.99949
10	1,439	−0.19	−0.40	0.92186	0.99935
11	1,535	6.55	0.03	0.77361	0.99510
12	1,789	5.12	−1.05	0.74005	0.99321
13	1,973	−5.80	−0.48	0.86214	0.99483
14	2,029	−12.08	0.14	0.00010	0.99272
15	2,379	4.69	0.23	0.87470	0.99142
16	2,989	7.87	−0.09	0.97710	0.99787
17	3,359	0.13	−0.75	0.64763	0.99639
18	3,446	3.26	−0.01	0.87485	0.99941
19	3,584	−3.03	−0.28	0.32500	0.99405
20	4,108	−6.23	−0.65	0.53907	0.99891
21	5,042	−6.36	0.55	0.92307	0.99937
22	5,803	15.01	−0.03	0.94874	0.99449
23	5,941	−3.27	−0.02	0.58108	0.99479
24	6,455	3.15	0.29	0.09391	0.99704
25	7,900	5.75	−0.23	0.94559	0.98560
26	8,959	5.19	0.69	0.99391	0.99874
27	10,950	4.42	−0.06	0.97930	0.99699
28	11,624	1.60	0.14	0.99737	0.99984
29	12,386	3.83	0.61	0.94103	0.98614
30	14,669	6.27	0.15	0.93723	0.99847

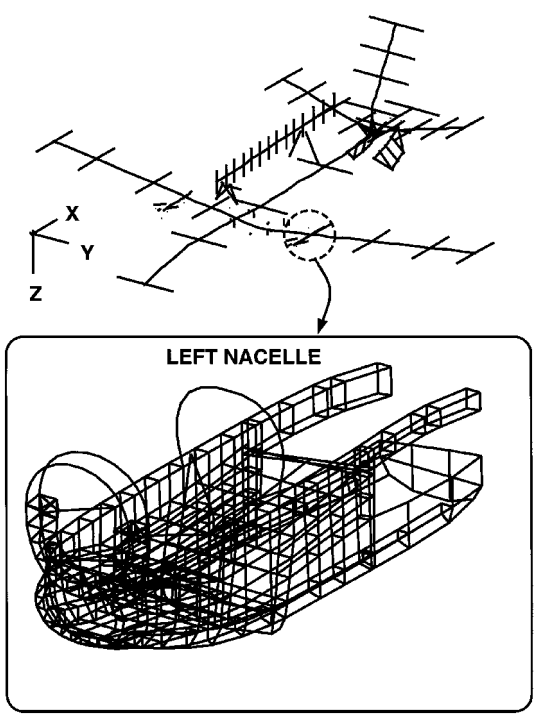

Fig. 4 Substructured FE model of SAAB 340AEW.

subspace. The subspace basis was not changed in this example; i.e., only a single step was taken in the outer loop. It took 135 iterations with the RQP method[20] to solve the minimization problem, which required almost 25 min of CPU time. Eigenvalues and *MAC* numbers of the initial and updated models are shown in Table 2. In the updated model, the eigenvalues are correct to within 1%, i.e., 0.5% error in cyclic frequency, and the worst *MAC* number is 0.986.

V. Conclusions

The model updating problem in structural dynamics is usually solved by iterative methods that require us to solve the analytical problem at least once in each iteration. Therefore, applications to large models are tedious and expensive. We suggest an approach where the size is reduced by projecting the problem onto a subspace. Although several choices of basis vectors for the subspace are possible, the eigenvectors of the full model are natural to use in a structural dynamic application and are, hence, used in our derivation. The proposed method can also handle substructured models; actually, substructuring is beneficial because it reduces the computational effort to project the model onto the selected subspace.

Utilizing the reduced problem to perform model updating, one achieves very short execution times, which makes it possible to handle large problems. Moreover, interactive design changes may be performed, so that the engineers' experience and intuition can be exploited, and insight for the particular design problem is gained. It is noted, however, that large design changes may require us to update the subspace basis. Therefore, the full model must always be solved subsequent to a subspace model correlation so as to check for this.

A small model problem and an industrial FE model are used to illustrate the performance of the suggested subspace method.

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